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RESEARCH ARTICLE

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Numerical solutions of doubly perturbed stochastic delay differential equations driven by Lèvy process

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Abstract In this paper, the numerical solutions of doubly perturbed stochastic delay differential equations driven by Lèvy process are investigated. Using the Euler–Maruyama method, we define the numerical solutions, and show that the numerical solutions converge to the true solutions under the local Lipschitz condition. As a corollary, we give the order of convergence under the global Lipschtiz condition.

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الملخص

في هذه الورقة، تمّت دراسة الحلول العددية لمعادلات تفاضلية، مضاعفة الاضطراب و ذات تأخير عشوائي، موجّهة بعملية ليفي. وباستخدام طريقة أويلر - ماروياما، نعرّف الحلول العددية، ونبين أن هذه الحلول العددية تتقارب إلى حلول حقيقية تحت شرط لبشتز المحلي. ونتيجة لذلك، نعطي معدّل التقارب تحت شرط لبشتز الشامل.

1 Introduction

The purpose of this paper is to study the numerical solutions of doubly perturbed stochastic delay differential equations (DPSDDEs) driven by Lèvy process. Such DPSDDEs take the form:

$$\begin{aligned} x(t) &= x(0) + \int_0^t f(x(s), x(\delta(s)))ds + \int_0^t g(x(s), x(\delta(s)))dB(s) \\ &\quad + \int_0^t \int_{|l|<c} H(x(s-), x(\delta(s)-), l)\widetilde{N}(ds, dl) + \alpha \sup_{0\le s\le t} x(s) + \beta \inf_{0\le s\le t} x(s), \end{aligned} \tag{1.1}$$

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where $0 < \alpha, \beta < 1$ and $\alpha + \beta < 1$; $B(t), t \geq 0$ is a standard Brownian motion and $\widetilde{N}(dt, dl)$ is a compensated Poisson random measure; the mappings $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ and $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are all Borel-measurable functions; $c \in (0, +\infty]$ is the maximum allowable jump size; $\delta(s)$ stands for the time delay.

The perturbed Brownian motion

$$x(t) = B(t) + \alpha \sup_{0\leq s\leq t} x(s) + \beta \inf_{0\leq s\leq t} x(s),$$

is a limit process from a "weak" polymers model of Norris et al. [14], and it also arises as the scaling limit of some self-interacting random walks (see e.g. [17,18]). This process behaves exactly as a Brownian motion except when it hit its past maximum (or minimum), where it gets an extra 'push'.

During the past 30 years, there are numerous papers concerned with Brownian motions perturbed at their extrema (see e.g. [2,4–6,15]). For example, Doney and Zhang [6] obtained the existence and uniqueness of the solutions for the following perturbed nonlinear diffusion process

$$x(t) = x(0) + \int_0^t f(s, x(s))ds + \int_0^t g(s, x(s))dB(s) + \alpha \sup_{0\leq s\leq t} x(s), \tag{1.2}$$

where f and g are Lipschitz continuous functions. Recently, the large deviation principle for Eq. (1.2) was established by Bo and Zhang [2].

Most of DPSDDEs do not have explicit solutions and hence require numerical solutions. However, there are few numerical methods available for DPSDDEs yet. The numerical methods for stochastic differential equations (SDEs) have been well studied (see e.g. [7,12,16]), but this is not the case for stochastic delay differential equations (SDDEs) as it has been pointed out in [8]. Buckwar [3] studied the numerical solution of SDDEs under the global Lipschitz condition, and Mao [13] investigated it under the local Lipschitz condition. Moreover, some results of the numerical solutions of SDEs with jumps were obtained. One can see Li and Chang [10], Wang and Gan [19].

However, few work has been done on the numerical solutions of DPSDDEs. In this paper, we define the numerical solutions by the Euler–Maruyama method, and show that the numerical solutions converge to the true solution under the local Lipschitz condition. As a corollary, we give the order of convergence under the global Lipschitz condition.

The paper is organized as follows. In Sect. 2, we introduce some necessary notations and define the Euler–Maruyama approximate solution to DPSDDEs. In Sect. 3, a number of useful lemmas are presented. In Sect. 4, the convergence of numerical solutions is proved in the sense of mean square. Finally, in Sect. 5, conclusions and discussions on future research topics are given.

2 Preliminary notation and Euler–Maruyama method

Throughout this paper, we let $(\Omega, \mathcal{F}, P)$ be a complete probability space with some filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., the filtration is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $|x|$ be the Euclidean norm of $x \in \mathbb{R}^d$. Assume $\tau > 0$ and $\mathbb{R}^+ = [0, +\infty)$. Let $C([-\tau, 0]; \mathbb{R}^d)$ be the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^d$ with norm $||\varphi|| = \sup_{-\tau\leq\theta\leq 0} \varphi(\theta)$. Denote by $L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^d)$ the family of $\mathcal{F}_t$-measurable, $C([-\tau, 0]; \mathbb{R}^d)$-valued random variables $\xi = \{\xi(s), -\tau \leq s \leq 0\}$ such that $E||\xi||^p = \sup_{-\tau\leq s\leq 0} E|\xi(s)|^p < +\infty$. Let $C(a)$ denote a constant, whose value depends only on a. For simplicity, we denote by $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

Let $B = (B(t), t \geq 0)$ be an m-dimensional standard $\mathcal{F}_t$-adapted Brownian motion and N be an independent $\mathcal{F}_t$-adapted Poisson random measure defined on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ with compensator $\widetilde{N}$ and intensity measure ν, where ν is a Lèvy measure so that $\widetilde{N}(dt, dy) := N(dt, dy) - \nu(dy)dt$ and $\int_{\mathbb{R}^d-\{0\}}(|y|^2 \wedge 1)\nu(dy) < \infty$. For Eq. (1.1), the initial data $x(0) = \xi(0) \in L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^d)$, and $\delta : [0, \infty) \to \mathbb{R}$ is a Lipschitz continuous function which satisfies

$$-\tau \leq \delta(t) \leq t \quad \text{and} \quad |\delta(t) - \delta(s)| \leq \rho|t - s|, \forall t, s \geq 0, \tag{2.1}$$

for some positive constant ρ.

In this paper, we make the following assumptions:

(H1) There exists a constant $K_1 > 0$ such that for all $-\tau \le s < t \le 0$

$$E|\xi(t) - \xi(s)|^2 \le K_1(t - s). \tag{2.2}$$

(H2) For each n and each $2 \le \eta \le p$, there exists a positive constant $K_2(n)$, such that

$$|f(x_1, y_1) - f(x_2, y_2)|^2 \vee |g(x_1, y_1) - g(x_2, y_2)|^2 \le K_2(n)(|x_1 - x_2|^2 + |y_1 - y_2|^2)$$

and

$$\int_{|l|<c} |H(x_1, y_1, l) - H(x_2, y_2, l)|^\eta \nu(dl) \le K_2(n)(|x_1 - x_2|^\eta + |y_1 - y_2|^\eta),$$

for $x_k, y_k \in \mathbb{R}^d$ with $|x_k| \vee |y_k| \le n, k = 1, 2$.

(H3) For $2 \le \eta \le p$, there exists a positive constant K_3 such that

$$|f(x, y)|^2 \vee |g(x, y)|^2 \le K_3(1 + |x|^2 + |y|^2) \tag{2.3}$$

and

$$\int_{|l|<c} |H(x, y, l)|^\eta \nu(dl) \le K_3(1 + |x|^\eta + |y|^\eta). \tag{2.4}$$

The existence and uniqueness of the solution for Eq. (1.1) can be guaranteed by (H2–H3) (see [11]). Now, we will define the Euler–Maruyama approximate solution of the DPSDDEs (1.1). Let the time-step size $\Delta \in (0, 1)$ be a fraction of τ, that is $\Delta = \frac{\tau}{N}$ for some sufficiently large integer N. Then the discrete Euler–Maruyama approximate solution is defined by

$$\begin{aligned} \bar{y}((k+1)\Delta) &= \bar{y}(k\Delta) + f(\bar{y}(k\Delta), \bar{y}(I_\Delta[\delta(k\Delta)]\Delta))\Delta + g(\bar{y}(k\Delta), \bar{y}(I_\Delta[\delta(k\Delta)]\Delta))\Delta B_k \\ &\quad + \int_{|l|<c} H(\bar{y}(k\Delta), \bar{y}(I_\Delta[\delta(k\Delta)]\Delta), l)\widetilde{N}(\Delta, dl) + \alpha \sup_{0\le j\le(k+1)} \bar{y}(j\Delta) + \beta \inf_{0\le j\le(k+1)} \bar{y}(j\Delta). \end{aligned} \tag{2.5}$$

with $\bar{y}(t) = \xi(t)$ on $-\tau \le t \le 0$. Here $I_\Delta[u]$ denotes the integer part of the real number $\frac{u}{\Delta}$. Thus, $I_\Delta[\delta(k\Delta)]$ represents the integer part of $\frac{\delta(k\Delta)}{\Delta}$. Clearly,

$$-\tau \le I_\Delta[\delta(k\Delta)]\Delta \le k\Delta \quad \text{for every } k \ge 0.$$

To define the continuous extension, let us introduce two step processes

$$z_1(t) = \sum_{k=0}^{\infty} I_{[k\Delta,(k+1)\Delta)}(t)\bar{y}(k\Delta),$$

$$z_2(t) = \sum_{k=0}^{\infty} I_{[k\Delta,(k+1)\Delta)}(t)\bar{y}(I_\Delta[\delta(k\Delta)]\Delta).$$

We define the continuous Euler–Maruyama approximate solution as follows

$$\begin{aligned} y(t) &= \xi(0) + \int_0^t f(z_1(s), z_2(s))ds + \int_0^t g(z_1(s), z_2(s))dB(s) + \int_0^t \int_{|l|<c} H(z_1(s), z_2(s), l)\widetilde{N}(ds, dl) \\ &\quad + \alpha \sup_{0\le s\le t} y_1(s, t) + \beta \inf_{0\le s\le t} y(s), \end{aligned} \tag{2.6}$$

where

$$y_1(s,t)=\sum_{j=0}^{I_\Delta[t]} I_{[j\Delta,(j+1)\Delta)}(s)\bar{y}(j\Delta)+I_{(I_\Delta[t]\Delta,t)}(s)y(s).$$

Let $y(t)=\xi(t)$, when $-\tau\le t\le 0$. Hence, for every $k\ge 0$ and $k\Delta\le t<(k+1)\Delta$, we have

$$\begin{aligned} y(t) &= y(k\Delta)+\int_{k\Delta}^{t} f(z_1(s),z_2(s))ds+\int_{k\Delta}^{t} g(z_1(s),z_2(s))dB(s)+\int_{k\Delta}^{t}\int_{|l|<c} H(z_1(s),z_2(s),l)\widetilde{N}(ds,dl)\\ &\quad+\alpha\sup_{0\le s\le t} y_1(s,t)+\beta\inf_{0\le s\le t} y(s)-\alpha\sup_{0\le j\le k}\bar{y}(j\Delta)-\beta\inf_{0\le j\le k}\bar{y}(j\Delta). \end{aligned}\tag{2.7}$$

Remark 2.1 We evaluate the jump integral by $\int_{|l|<c} H(\bar{y}(k\Delta),\bar{y}(I_\Delta[\delta(k\Delta)]\Delta),l)\widetilde{N}(\Delta,dl)$, where $\widetilde{N}(\Delta,dy)=N(\Delta,dy)-\nu(dy)\Delta$. In Theorem 4.1, we shall show that the error of Euler–Maruyama approximate solution converges to zero in L^2 as $\Delta\to 0$.

3 Lemmas and corollary

The key contribution of this paper is to show that the Euler–Maruyama approximate solutions will converge to the true solutions of Eq. (1.1) under the local Lipschitz condition. The proof of the result is rather technical, so we present several lemmas before the main result.

Lemma 3.1 *Assume that* Eq. (2.4) *holds. Then there exists a constant* $C_{p,T}>0$, *such that for* $p\ge 2$,

$$E\left[\sup_{0\le s\le t}\left|\int_0^s\int_{|l|<c} H(x(u-),x(\delta(u)-),l)\widetilde{N}(du,dl)\right|^p\right]\le C_{p,T}+C_{p,T}\int_0^t E\left[\sup_{-\tau\le u\le s}|x(u)|^p\right]ds,\ t\in[0,T].$$

Proof By Theorem 2.11 in [9] (or Theorem 2.4 in [1]), there exists a positive constant $C(p)$, such that

$$\begin{aligned} &E\left[\sup_{0\le s\le t}\left|\int_0^s\int_{|l|<c} H(x(u-),x(\delta(u)-),l)\widetilde{N}(du,dl)\right|^p\right]\\ &\le C(p)\left[E\int_0^t\int_{|l|<c}|H(x(s-),x(\delta(s)-),l)|^p\nu(dl)ds+E\left(\int_0^t\int_{|l|<c}|H(x(s-),x(\delta(s)-),l)|^2\nu(dl)ds\right)^{\frac{p}{2}}\right]. \end{aligned}$$

In view of Eq. (2.4), we obtain

$$\begin{aligned} &C(p)\left[E\int_0^t\int_{|l|<c}|H(x(s-),x(\delta(s)-),l)|^p\nu(dl)ds+E\left(\int_0^t\int_{|l|<c}|H(x(s-),x(\delta(s)-),l)|^2\nu(dl)ds\right)^{\frac{p}{2}}\right]\\ &\le C(p)\left\{K_3E\int_0^t[1+|x(s-)|^p+|x(\delta(s)-)|^p]ds+T^{\frac{p}{2}-1}K_3^{\frac{p}{2}}E\int_0^t[1+|x(s-)|^2+|x(\delta(s)-)|^2]^{\frac{p}{2}}ds\right\} \end{aligned}$$

$$\leq C(p)K_3T + 2C(p)K_3 \int_0^t E\left[\sup_{-\tau\leq u\leq s} |x(u)|^p\right] ds + 2^{\frac{p-2}{2}} K_3^{\frac{p}{2}} T^{\frac{p}{2}} + 2^{p-1} K_3^{\frac{p}{2}} T^{\frac{p}{2}-1} \int_0^t E\left[\sup_{-\tau\leq u\leq s} |x(u)|^p\right] ds$$

$$\leq C_{p,T} + C_{p,T} \int_0^t E\left[\sup_{-\tau\leq u\leq s} |x(u)|^p\right] ds,$$

where $C_{p,T} = [C(p)K_3T + 2^{\frac{p-2}{2}} K_3^{\frac{p}{2}} T^{\frac{p}{2}}] \vee [2C(p)K_3 + 2^{p-1} K_3^{\frac{p}{2}} T^{\frac{p}{2}-1}]$. This completes the proof. □

Lemma 3.2 *Under condition* (H3)*, for every* $p \geq 2$*, there exists a positive constant* C_1*, such that*

$$E\left(\sup_{-\tau\leq s\leq T} |x(s)|^p\right) \leq C_1.$$

Proof For any $0 \leq s \leq t \leq T$, we get from Eq. (1.1),

$$\begin{aligned}\sup_{0\leq s\leq t} |x(s)| \leq \sup_{0\leq s\leq t} \Bigg| \xi(0) &+ \int_0^s f(x(u), x(\delta(u)))du + \int_0^s g(x(u), x(\delta(u)))dB(u) \\ &+ \int_0^s \int_{|l|<c} H(x(u-), x(\delta(u)-), l)\widetilde{N}(du, dl) \Bigg| + (\alpha+\beta) \sup_{0\leq s\leq t} |x(s)|,\end{aligned}$$

which yields

$$\begin{aligned}\sup_{0\leq s\leq t} |x(s)| \leq \sup_{0\leq s\leq t} \frac{1}{1-\alpha-\beta} \Bigg| \xi(0) &+ \int_0^s f(x(u), x(\delta(u)))du + \int_0^s g(x(u), x(\delta(u)))dB(u) \\ &+ \int_0^s \int_{|l|<c} H(x(u-), x(\delta(u)-), l)\widetilde{N}(du, dl) \Bigg|.\end{aligned}$$

Therefore

$$\begin{aligned}E\left(\sup_{0\leq s\leq t} |x(s)|^p\right) \leq \frac{1}{(1-\alpha-\beta)^p} E\Bigg[\sup_{0\leq s\leq t} \Bigg| \xi(0) &+ \int_0^s f(x(u), x(\delta(u)))du \\ &+ \int_0^s g(x(u), x(\delta(u)))dB(u) + \int_0^s \int_{|l|<c} H(x(u-), x(\delta(u)-), l)\widetilde{N}(du, dl) \Bigg|^p\Bigg].\end{aligned}$$

Thanks to inequality $|a+b+c+d| \leq 4^{p-1}(|a|^p + |b|^p + |c|^p + |d|^p)$, we deduce that

$$\begin{aligned}&E\left(\sup_{0\leq s\leq t} |x(s)|^p\right) \\ &\leq C_{1,1}\Bigg\{E|\xi(0)|^p + E\left[\sup_{0\leq s\leq t}\left|\int_0^s f(x(u), x(\delta(u)))du\right|^p\right] + E\left[\sup_{0\leq s\leq t}\left|\int_0^s g(x(u), x(\delta(u)))dB(u)\right|^p\right] \\ &\quad + E\left[\sup_{0\leq s\leq t}\left|\int_0^s \int_{|l|<c} H(x(u-), x(\delta(u)-), l)\widetilde{N}(du, dl)\right|^p\right]\Bigg\}, \qquad (3.1)\end{aligned}$$

where $C_{1,1} = \frac{4^{p-1}}{(1-\alpha-\beta)^p}$. The Hölder inequality and (H3) imply that

$$
\begin{aligned}
&E\left[\sup_{0\le s\le t}\left|\int_0^s f(x(u),x(\delta(u)))du\right|^p\right]\\
&\le T^{p-1}E\int_0^t |f(x(u),x(\delta(u)))|^p du\\
&\le T^{p-1}E\int_0^t K_3^{\frac{p}{2}}(1+|x(s)|^2+|x(\delta(s))|^2)^{\frac{p}{2}}ds\\
&\le T^{p-1}E\int_0^t K_3^{\frac{p}{2}}(1+2\sup_{-\tau\le u\le s}|x(u)|^2|)^{\frac{p}{2}}ds\\
&\le 2^{\frac{p-2}{2}}K_3^{\frac{p}{2}}T^p+2^{p-1}K_3^{\frac{p}{2}}T^{p-1}\int_0^t E\left[\sup_{-\tau\le u\le s}|x(u)|^p\right]ds. \qquad (3.2)
\end{aligned}
$$

On the other hand, by virtue of the Burkholder–Davis–Gundy inequality (Theorem 1.7.3 in [12]) and (H3), we have

$$
\begin{aligned}
&E\left[\sup_{0\le s\le t}\left|\int_0^s g(x(u),x(\delta(u)))dB(u)\right|^p\right]\\
&\le\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}}E\left[\int_0^t |g(x(u),x(\delta(u)))|^2 ds\right]^{\frac{p}{2}}\\
&\le\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}}T^{\frac{p-2}{2}}E\int_0^t |g(x(u),x(\delta(u)))|^p ds\\
&\le 2^{\frac{p-2}{2}}K_3^{\frac{p}{2}}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}}T^{\frac{p}{2}}+2^{p-1}K_3^{\frac{p}{2}}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}}T^{\frac{p-2}{2}}\int_0^t E\left[\sup_{-\tau\le u\le s}|x(u)|^p\right]ds. \qquad (3.3)
\end{aligned}
$$

It follows from (3.1–3.3) and Lemma 4.1 that

$$E\left(\sup_{0\le s\le t}|x(s)|^p\right)\le C_{1,2}+C_{1,1}E||\xi||^p+C_{1,3}E\left[\sup_{-\tau\le u\le s}|x(u)|^p\right]ds, \qquad (3.4)$$

where $C_{1,2}=C_{1,1}\left[2^{\frac{p-2}{2}}K_3^{\frac{p}{2}}T^p+2^{\frac{p-2}{2}}K_3^{\frac{p}{2}}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}}T^{\frac{p}{2}}+C_{p,T}\right]$, $C_{1,3}=C_{1,1}\left[2^{p-1}K_3^{\frac{p}{2}}T^{p-1}+2^{p-1}K_3^{\frac{p}{2}}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}}T^{\frac{p-2}{2}}+C_{p,T}\right]$. Notice that

$$E\left(\sup_{-\tau\le s\le t}|x(s)|^p\right)\le E||\xi||^p\vee E\left(\sup_{0\le s\le t}|x(s)|^p\right),$$

we get

$$E\left(\sup_{-\tau\le s\le t}|x(s)|^p\right)\le C_{1,2}+C_{1,1}E||\xi||^p+C_{1,3}\int_0^t E\left[\sup_{-\tau\le u\le s}|x(u)|^p\right]ds.$$

Combining this with the Gronwall's inequality, we get

$$E\left(\sup_{-\tau\le s\le t}|x(s)|^p\right)\le[C_{1,2}+C_{1,1}E||\xi||^p]e^{C_{1,3}T}.$$

This implies our claim immediately by letting

$$C_1=[C_{1,2}+C_{1,1}E||\xi||^p]e^{C_{1,3}T}$$

and the lemma follows. □

Lemma 3.3 *Under condition* (H3)*, for any $p\ge 2$, there exists a positive constant C_2, such that*

$$E\left(\sup_{-\tau\le s\le T}|y(s)|^p\right)\le C_2.$$

Proof The proof is similar to Lemma 3.2, here we omit it. □

For each $n>0$, we define the stopping times

$$\tau_n:=\inf\{t\ge 0:|x(t)|\ge n\},\quad \sigma_n:=\inf\{t\ge 0:|y(t)|\ge n\}$$

and $v_n=\tau_n\wedge\sigma_n$. (We set $\inf\emptyset=\infty$).

Corollary 3.1 *Under condition* (H2)*, there exists a positive constant $C_2(n)$, such that*

$$E\left[\sup_{-\tau\le s\le T}|y(s\wedge v_n)|^2\right]\le C_2(n).$$

Proof From (H2), for every $n>0$ and $|x|\vee|y|\le n$, we have

$$\begin{aligned}|f(x,y)|^2&\le 2|f(x,y)-f(0,0)|^2+2|f(0,0)|^2\le 2K_2(n)(|x|^2+|y|^2)+2|f(0,0)|^2\\&\le K_4(n)(1+|x|^2+|y|^2).\end{aligned}$$

Similarly, we have

$$|g(x,y)|^2\le K_4(n)(1+|x|^2+|y|^2)$$

and

$$\int_{|l|<c}|H(x,y,l)|^\eta\nu(dl)\le K_4(n)(1+|x|^\eta+|y|^\eta),\tag{3.5}$$

where $K_4(n)=2^{\eta-1}(K_2(n)\vee|f(0,0)|^2\vee|g(0,0)|^2\vee\int_{|l|<c}|H(0,0,l)|^\eta\nu(dl)),2\le\eta\le p$. Thus the corollary follows from Lemma 3.3. □

Lemma 3.4 *Under the condition* (H2)*, for any $t\in[0,T]$, we have*

$$E|y(t\wedge v_n)-z_1(t\wedge v_n)|^2\le C_3(n)\Delta.$$

Proof Clearly, for any $t\in[0,T]$, we may choose a positive integer k such that $t\wedge v_n\in[k\Delta,(k+1)\Delta)$ and $k=k(\omega)$ is dependent on the sample path. It follows that

$$y(t\wedge v_n)-z_1(t\wedge v_n)=y(t\wedge v_n)-\bar{y}(k\Delta).$$

From Eq. (2.7), we have

$$\begin{aligned}&y(t\wedge v_n)-\bar{y}(k\Delta)\\&\le\int_{k\Delta}^{t\wedge v_n}f(z_1(s),z_2(s))ds+\int_{k\Delta}^{t\wedge v_n}g(z_1(s),z_2(s))dB(s)+\int_{k\Delta}^{t\wedge v_n}\int_{|l|<c}H(z_1(s),z_2(s),l)\widetilde{N}(ds,dl)\\&\quad+\alpha\sup_{0\le s\le t\wedge v_n}y_1(s,t\wedge v_n)+\beta\inf_{0\le s\le t\wedge v_n}y(s)-\alpha\sup_{0\le j\le k}\bar{y}(j\Delta)-\beta\inf_{0\le j\le k}\bar{y}(j\Delta).\end{aligned}\tag{3.6}$$

Since $\inf_{0\le s\le t\wedge v_n} y(s) \le \inf_{0\le j\le k} \bar{y}(j\Delta)$, we get from Eq. (3.6) that

$$\begin{aligned}
&|y(t\wedge v_n) - \bar{y}(k\Delta)| \\
&\quad\le \left| \int_{k\Delta}^{t\wedge v_n} f(z_1(s), z_2(s))ds + \int_{k\Delta}^{t\wedge v_n} g(z_1(s), z_2(s))dB(s) + \int_{k\Delta}^{t\wedge v_n}\int_{|l|<c} H(z_1(s), z_2(s), l)\widetilde{N}(ds, dl) \right| \\
&\qquad + \alpha \left| \sup_{0\le s\le t\wedge v_n} y_1(s, t\wedge v_n) - \sup_{0\le j\le k} \bar{y}(j\Delta) \right|.
\end{aligned}$$

Noting that $\sup_{0\le s\le t\wedge v_n} y_1(s, t\wedge v_n) = \sup_{1\le j\le k} \bar{y}(j\Delta) \vee \sup_{k\Delta\le s\le t\wedge v_n} y(s)$, we derive

$$\left| \sup_{0\le s\le t\wedge v_n} y_1(s, t\wedge v_n) - \sup_{0\le j\le k} \bar{y}(j\Delta) \right| \le \sup_{k\Delta\le s\le t\wedge v_n} |y(s) - \bar{y}(k\Delta)|. \tag{3.7}$$

In fact, if $\sup_{0\le s\le t\wedge v_n} y_1(s, t\wedge v_n) = \sup_{1\le j\le k} \bar{y}(j\Delta)$, (3.7) obviously holds, and if

$$\sup_{0\le s\le t\wedge v_n} y_1(s, t\wedge v_n) = \sup_{k\Delta\le s\le t\wedge v_n} y(s),$$

we have

$$\sup_{0\le s\le t\wedge v_n} y_1(s, t\wedge v_n) - \sup_{0\le j\le k} \bar{y}(j\Delta) \le \sup_{k\Delta\le s\le t\wedge v_n} y(s) - \bar{y}(k\Delta) \le \sup_{k\Delta\le s\le t\wedge v_n} |y(s) - \bar{y}(k\Delta)|.$$

Thus, we see that

$$\begin{aligned}
\sup_{k\Delta\le s\le t\wedge v_n} |y(s) - \bar{y}(k\Delta)| \;\le\; &\sup_{k\Delta\le s\le t\wedge v_n} \left| \int_{k\Delta}^{s} f(z_1(u), z_2(u))du + \int_{k\Delta}^{s} g(z_1(u), z_2(u))dB(u) \right. \\
&\left. + \int_{k\Delta}^{s}\int_{|l|<c} H(z_1(u), z_2(u), l)\widetilde{N}(du, dl) \right| + \alpha \sup_{k\Delta\le s\le t\wedge v_n} |y(s) - \bar{y}(k\Delta)|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\sup_{k\Delta\le s\le t\wedge v_n} |y(s) - \bar{y}(k\Delta)| \;\le\; &\sup_{k\Delta\le s\le t\wedge v_n} \frac{1}{1-\alpha} \left| \int_{k\Delta}^{s} f(z_1(u), z_2(u))du + \int_{k\Delta}^{s} g(z_1(u), z_2(u))dB(u) \right. \\
&\left. + \int_{k\Delta}^{s}\int_{|l|<c} H(z_1(u), z_2(u), l)\widetilde{N}(du, dl) \right|.
\end{aligned}$$

Combining this with inequality $|a+b+c|^2 \le 3(|a|^2 + |b|^2 + |c|^2)$, one can get

$$\begin{aligned}
&E\left[\sup_{k\Delta\le s\le t\wedge v_n} |y(s) - \bar{y}(k\Delta)|^2 \right] \\
&\quad\le 3C_{3,1} \left\{ E\left[\sup_{k\Delta\le s\le t\wedge v_n} \left| \int_{k\Delta}^{s} f(z_1(u), z_2(u))du \right|^2 \right] + E\left[\sup_{k\Delta\le s\le t\wedge v_n} \left| \int_{k\Delta}^{s} g(z_1(u), z_2(u))dB(u) \right|^2 \right] \right. \\
&\qquad \left. + E\left[\sup_{k\Delta\le s\le t\wedge v_n} \left| \int_{k\Delta}^{s}\int_{|l|<c} H(z_1(u), z_2(u), l)\widetilde{N}(du, dl) \right|^2 \right] \right\},
\end{aligned} \tag{3.8}$$

where $C_{3,1} = \frac{1}{(1-\alpha)^2}$. Using Doob's martingale inequality, Itô's isometry property and Eq. (3.5) yields that

$$E\left[\sup_{k\Delta \le s \le t\wedge v_n}\left|\int_{k\Delta}^{s}\int_{|l|<c} H(z_1(s), z_2(s), l)\widetilde{N}(du, dl)\right|^2\right]$$
$$\le 4E\int_{k\Delta}^{t\wedge v_n}\int_{|l|<c} |H(z_1(u), z_2(u), l)|^2\, ds\, v(dl)$$
$$\le 4K_4(n)\int_{k\Delta}^{t\wedge v_n}\left(1 + 2E\left[\sup_{-\tau\le u\le s}|y(u)|^2\right]\right)ds. \tag{3.9}$$

Substituting Eqs. (3.2) and (3.3) with $p = 2$, $K_3 = K_4(n)$ and Eq. (3.9) into (3.8), we get

$$E\left[\sup_{k\Delta \le s \le t\wedge v_n}|y(s) - \bar{y}(k\Delta)|^2\right]$$
$$\le 3C_{3,1}K_4(n)(T+8)\int_{k\Delta}^{t\wedge v_n}\left(1 + 2E\left[\sup_{-\tau\le u\le s}|y(u)|^2\right]\right)ds. \tag{3.10}$$

It follows from Eq. (3.10) and Corollary 3.1 that

$$E\left[\sup_{k\Delta \le s \le t\wedge v_n}|y(s) - \bar{y}(k\Delta)|^2\right] \le 3C_{3,1}K_4(n)(T+8)\int_{k\Delta}^{t\wedge v_n}(1 + 2C_2(n))ds$$
$$\le 3C_{3,1}K_4(n)(T+8)(1 + 2C_2(n))\Delta.$$

Therefore,

$$E|y(t\wedge v_n) - z_1(t\wedge v_n)|^2 = E|y(t\wedge v_n) - \bar{y}(k\Delta)|^2 \le E\sup_{k\Delta\le s\le t}|y(s\wedge v_n) - \bar{y}(k\Delta)|^2 \le C_3(n)\Delta \tag{3.11}$$

where $C_3(n) = 3C_{3,1}K_4(n)(T+8)(1 + 2C_2(n))$. This completes the proof. □

Corollary 3.2 *Under condition* (H2)*, we have*

$$E\sup_{0\le t\le T}|y(t) - z_1(t)|^2 \le C_3(n)\Delta.$$

Proof For any $t \in [0, T]$, we can choose a positive integer k such that $t \in [k\Delta, (k+1)\Delta)$ and $k = k(\omega)$ is dependent on the sample path. Therefore

$$y(t) - z_1(t) = y(t) - \bar{y}(k\Delta).$$

By Eq. (3.11), we have

$$E\sup_{k\Delta\le t<(k+1)\Delta}|y(t) - z_1(t)|^2 = E\sup_{k\Delta\le t<(k+1)\Delta}|y(t) - \bar{y}(k\Delta)|^2 \le C_3(n)\Delta,$$

where $C_3(n)$ is independent of k. Thus, for any k, we can get $E\sup_{k\Delta\le t<(k+1)\Delta}|y(t) - z_1(t)|^2 \le C_3(n)\Delta$ and the Corollary follows. □

Lemma 3.5 *Under* (H1) *and* (H2)*, if* Δ *is small enough such that* $(\rho + 1)\Delta \le 1$*, then there exists a positive constant* $C_4(n)$*, such that*

$$E|y(\delta(t\wedge v_n)) - z_2(t\wedge v_n)|^2 \le C_4(n)\Delta,\quad \forall t \in [0, T].$$

Proof One just needs to repeat the proof of the Lemma 3.3 as in Mao [13], so we omit the detailed proof. □

4 Main result

The primary aim of this paper is to establish the following main result.

Theorem 4.1 *Under hypotheses* (H1–H3)*, the approximate solution* (2.6) *converges to the true solution of* Eq. (1.1) *in the sense*

$$\lim_{\Delta\to 0} E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2\right]=0.$$

Proof It is obvious that

$$\begin{aligned}&E\sup_{0\le t\le T}|x(t)-y(t)|^2\\&\le E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2 I_{\{\tau_n>T \text{ and } \sigma_n>T\}}\right]+E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2 I_{\{\tau_n\le T \text{ or } \sigma_n\le T\}}\right].\end{aligned}\tag{4.1}$$

By the Young inequality

$$ab\le 2h\frac{a^{\frac{p}{2}}}{p}+\frac{p-2}{ph^{\frac{2}{p-2}}}b^{\frac{p}{p-2}},$$

where $a, b, h > 0$ and $p > 2$, we obtain

$$E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2 I_{\{\tau_n\le T \text{ or } \sigma_n\le T\}}\right]\le E\left[\frac{2h}{p}\sup_{0\le t\le T}|x(t)-y(t)|^p\right]+E\left[\frac{p-2}{p}\frac{1}{h^{\frac{2}{p-2}}}I_{\{\tau_n\le T \text{ or } \sigma_n\le T\}}\right].$$

Consequently,

$$E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2 I_{\{\tau_n\le T \text{ or } \sigma_n\le T\}}\right]\le \frac{2h}{p}E\left[\sup_{0\le t\le T}|x(t)-y(t)|^p\right]+\frac{p-2}{p}\frac{1}{h^{\frac{2}{p-2}}}P(\tau_n\le T \text{ or } \sigma_n\le T).$$

Set $C_5 = C_1 \vee C_2$. Notice that

$$P(\tau_n\le T)\le E\left[I_{\{\tau_n\le T\}}\frac{|x(\tau_n)|^p}{n^p}\right]\le\frac{1}{n^p}E\left[\sup_{-\tau\le t\le T}|x(t)|^p\right]\le\frac{C_5}{n^p}$$

and

$$P(\sigma_n\le T)\le\frac{C_5}{n^p}.$$

We get

$$P(\tau_n\le T \text{ or } \sigma_n\le T)\le P(\tau_n\le T)+P(\sigma_n\le T)\le\frac{2C_5}{n^p}.$$

Moreover,

$$E\left[\sup_{0\le t\le T}|x(t)-y(t)|^p\right]\le 2^{p-1}E\left[\sup_{0\le t\le T}|x(t)|^p+\sup_{0\le t\le T}|y(t)|^p\right]\le 2^pC_5.$$

Substituting these inequalities above into Eq. (4.1) leads to

$$E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2\right]\le E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2 I_{\{\tau_n>T \text{ and } \sigma_n>T\}}\right]+\frac{2^{p+1}hC_5}{p}+\frac{2(p-2)C_5}{ph^{\frac{2}{p-2}}n^p}.\tag{4.2}$$

Let us now estimate the first term on the right-hand side of Eq. (4.2). Clearly,

$$\begin{aligned} &E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2 I_{\{\tau_n>T \text{ and } \sigma_n>T\}}\right]\\ &\quad=E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2 1_{\{v_n>T\}}\right]\le E\left[\sup_{0\le t\le T}|x(t\wedge v_n)-y(t\wedge v_n)|^2\right]. \end{aligned} \tag{4.3}$$

Thanks to Eqs. (1.1) and (2.6), we derive

$$\begin{aligned} &|x(t\wedge v_n)-y(t\wedge v_n)|\\ &\quad\le \Big|\int_0^{t\wedge v_n}[f(x(s),x(\delta(s)))-f(z_1(s),z_2(s))]ds+\int_0^{t\wedge v_n}[g(x(s),x(\delta(s)))-g(z_1(s),z_2(s))]dB(s)\\ &\qquad+\int_0^{t\wedge v_n}\int_{|l|<c}[H(x(s-),x(\delta(s)-),l)-H(z_1(s),z_2(s),l)]\widetilde{N}(ds,dl)\Big|+\beta\Big|\inf_{0\le s\le t\wedge v_n}x(s)-y(s)\Big|\\ &\qquad+\beta\Big|y(s)-\inf_{0\le s\le t\wedge v_n}y_1(s,t\wedge v_n)\Big|+\alpha\Big|\sup_{0\le s\le t\wedge v_n}x(s)-\sup_{0\le s\le t\wedge v_n}y_1(s,t\wedge v_n)\Big|, \end{aligned} \tag{4.4}$$

for $0\le t\wedge v_n\le T$. Obviously,

$$\left|\inf_{0\le s\le t\wedge v_n}x(s)-\inf_{0\le s\le t\wedge v_n}y(s)\right|\le\sup_{0\le s\le t\wedge v_n}|x(s)-y(s)|\,. \tag{4.5}$$

By the definition of $y_1(t)$, we have

$$\left|y(s)-\inf_{0\le s\le t\wedge v_n}y_1(s,t\wedge v_n)\right|\le|y(s)-y_1(s,t\wedge v_n)|=|y(s)-z_1(s)|I_{[0,I_\Delta[t\wedge v_n\wedge T]]}(s). \tag{4.6}$$

Noting that $\sup_{0\le s\le t\wedge v_n}y_1(s,t\wedge v_n)\le\sup_{0\le s\le t\wedge v_n}y(s)$, we can get

$$\begin{aligned} \sup_{0\le s\le t\wedge v_n}y_1(s,t\wedge v_n)-\sup_{0\le s\le t\wedge v_n}x(s)&\le\sup_{0\le s\le t\wedge v_n}y(s)-\sup_{0\le s\le t\wedge v_n}x(s)\\ &\le\left|\sup_{0\le s\le t\wedge v_n}x(s)-\sup_{0\le s\le t\wedge v_n}y(s)\right|\\ &\le\sup_{0\le s\le t\wedge v_n}|x(s)-y(s)|, \end{aligned}$$

which implies

$$\left|\sup_{0\le s\le t\wedge v_n}x(s)-\sup_{0\le s\le t\wedge v_n}y_1(s,t\wedge v_n)\right|\le\sup_{0\le s\le t\wedge v_n}|x(s)-y(s)|. \tag{4.7}$$

Substituting Eqs. (4.5), (4.6) and (4.7) into Eq. (4.4) gives

$$\begin{aligned} &|x(s\wedge v_n)-y(s\wedge v_n)|\\ &\quad\le\Bigg|\int_0^{s\wedge v_n}[f(x(u),x(\delta(u)))-f(z_1(u),z_2(u))]du+\int_0^{s\wedge v_n}[g(x(u),x(\delta(u)))-g(z_1(u),z_2(u))]dB(u)\\ &\qquad+\int_0^{s\wedge v_n}\int_{|l|<c}[H(x(u-),x(\delta(u)-),l)-H(z_1(u),z_2(u),l)]\widetilde{N}(du,dl)\}\Bigg|\\ &\qquad+(\alpha+\beta)\sup_{0\le s\le t\wedge v_n}|x(s)-y(s)|+|y(s)-z_1(s)|I_{[0,I_\Delta[t\wedge v_n\wedge T]]}(s). \end{aligned}$$

This proves that

$$\begin{aligned}
&\sup_{0\le s\le t\wedge v_n} |x(s\wedge v_n)-y(s\wedge v_n)|\\
&\le \sup_{0\le s\le t\wedge v_n}\Bigg|\int_0^s [f(x(u),x(\delta(u)))-f(z_1(u),z_2(u))]du+\int_0^s [g(x(u),x(\delta(u)))-g(z_1(u),z_2(u))]dB(u)\\
&\quad+\int_0^s\int_{|l|<c}[H(x(u-),x(\delta(u)-),l)-H(z_1(u),z_2(u),l)]\widetilde{N}(du,dl)\}\Bigg|\\
&\quad+(\alpha+\beta)\sup_{0\le s\le t\wedge v_n}\Bigg|x(s)-y(s)|+\sup_{0\le s\le T}|y(s)-z_1(s)\Bigg|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sup_{0\le s\le t\wedge v_n} |x(s)-y(s)|\\
&\le \frac{1}{1-\alpha-\beta}\sup_{0\le s\le t\wedge v_n}\Bigg|\int_0^s [f(x(u),x(\delta(u)))-f(z_1(u),z_2(u))]du\\
&\quad+\int_0^s [g(x(u),x(\delta(u)))-g(z_1(u),z_2(u))]dB(u)\\
&\quad+\int_0^s\int_{|l|<c}[H(x(u-),x(\delta(u)-),l)-H(z_1(u),z_2(u),l)]\widetilde{N}(du,dl)\Bigg|+\frac{1}{1-\alpha-\beta}\sup_{0\le s\le T}|y(s)-z_1(s)|.
\end{aligned}$$

Using the Doob martingale inequality, the local Lipschitz condition (H2) and Corollary 3.2 gives

$$\begin{aligned}
&E\left[\sup_{0\le s\le t\wedge v_n}|x(s)-y(s)|^2\right]\\
&\le \frac{3}{1-\alpha-\beta}\Bigg\{E\sup_{0\le s\le t\wedge v_n}\left|\int_0^s [f(x(u),x(\delta(u)))-f(z_1(u),z_2(u))]du\right|^2\\
&\quad+E\sup_{0\le s\le t\wedge v_n}\left|\int_0^s [g(x(u),x(\delta(u)))-g(z_1(u),z_2(u))]dB(u)\right|^2\\
&\quad+E\sup_{0\le s\le t\wedge v_n}\left|\int_0^s\int_{|l|<c}[H(x(u-),x(\delta(u)-),l)-H(z_1(u),z_2(u),l)]\widetilde{N}(du,dl)\right|^2\Bigg\}+\frac{C_3(n)}{1-\alpha-\beta}\Delta\\
&\le \frac{3K_2(n)(T+4)}{1-\alpha-\beta}E\left[\int_0^{t\wedge v_n}(|x(s)-z_1(s)|^2+|x(\delta(s))-z_2(s)|^2)ds\right]\\
&\quad+\frac{12K_2(n)}{1-\alpha-\beta}E\left[\int_0^{t\wedge v_n}(|x(s-)-z_1(s)|^2+|x(\delta(s)-)-z_2(s)|^2)ds\right]+\frac{C_3(n)}{1-\alpha-\beta}\Delta.
\end{aligned}$$

It follows that

$$E\left[\sup_{0\le s\le t\wedge v_n}|x(s)-y(s)|^2\right]\le\frac{6K_2(n)(T+4)}{1-\alpha-\beta}$$
$$\times E\left[\int_0^{t\wedge v_n}(|x(s)-y(s)|^2+|y(s)-z_1(s)|^2+|x(\delta(s))-y(\delta(s))|^2+|y(\delta(s))-z_2(s)|^2)ds\right]$$
$$+\frac{24K_2(n)}{1-\alpha-\beta}E\left[\int_0^{t\wedge v_n}(|x(s-)-y(s)|^2+|y(s)-z_1(s)|^2+|x(\delta(s)-)-y(\delta(s))|^2+|y(\delta(s))-z_2(s)|^2)ds\right]$$
$$+\frac{C_3(n)}{1-\alpha-\beta}\Delta.$$

Note that $y(t)=\xi(t)$ for any $t\in[-\tau,0]$ and

$$|x(s-)-y(s)|^2+|x(\delta(s)-)-y(\delta(s))|^2\le 2\sup_{0\le u\le s}|x(u)-y(u)|^2.$$

We get

$$E\left[\sup_{0\le s\le t\wedge v_n}|x(s)-y(s)|^2\right]$$
$$\le\frac{6K_2(n)(T+8)}{1-\alpha-\beta}\left\{2E\left[\int_0^{t\wedge v_n}\left(\sup_{0\le u\le s}|x(u)-y(u)|^2\right)ds\right]+E\left[\int_0^{t\wedge v_n}(|y(s)-z_1(s)|^2+|y(\delta(s))-z_2(s)|^2)ds\right]\right\}$$
$$+\frac{C_3(n)}{1-\alpha-\beta}\Delta.$$

Combining this with Lemmas 3.4 and 3.5, we see that

$$E\left[\sup_{0\le s\le t\wedge v_n}|x(s)-y(s)|^2\right]$$
$$\le C_{5,1}(n)\int_0^{t\wedge v_n}E\left[\sup_{0\le u\le s}|x(u)-y(u)|^2\right]ds+C_{5,2}(n)\Delta,$$

where $C_{5,1}(n)=\frac{12K_2(n)(T+8)}{1-\alpha-\beta}$, $C_{5,2}(n)=\frac{6K_2(n)(T+8)}{1-\alpha-\beta}(C_3(n)+C_4(n))T+\frac{C_3(n)}{1-\alpha-\beta}$. Then by the Gronwall inequality, we derive

$$E\left[\sup_{0\le s\le t\wedge v_n}|x(s)-y(s)|^2\right]\le C_{5,2}(n)e^{C_{5,1}(n)T}\Delta. \tag{4.8}$$

Thus, by Eqs. (4.2), (4.3) and (4.8) we get

$$E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2\right]\le C_{5,2}(n)e^{C_{5,1}(n)T}\Delta+\frac{2^{p+1}hC_5}{p}+\frac{2(p-2)C_5}{ph^{\frac{2}{p-2}}n^p}. \tag{4.9}$$

Now, for any $\epsilon>0$, we can choose a sufficiently small h such that $\frac{2^{p+1}hC_5}{p}<\frac{\epsilon}{3}$, then choose n so large that $\frac{2(p-2)C_5}{ph^{\frac{2}{p-2}}n^p}<\frac{\epsilon}{3}$ and finally choose Δ sufficiently small such that $C_{5,2}(n)e^{C_{5,1}(n)T}\Delta<\frac{\epsilon}{3}$. So

$$E\left[\sup_{0\le t\le T}|x(t)-y(t)|^2\right]\le\epsilon.$$

This completes the proof. □

Theorem 4.1 shows that under conditions (H1–H3) the Euler–Maruyama approximate solutions strongly converge to the true solutions. However, Theorem 4.1 does not give the order of the convergence. In the following, we will reveal the order of convergence, here we need the global Lipschitz condition instead of the local Lipschitz condition as follows.

(H4) For $2 \le \eta \le p$, there exists a positive constant K_2, such that

$$|f(x_1, y_1) - f(x_2, y_2)|^2 \vee |g(x_1, y_1) - g(x_2, y_2)|^2 \le K_2(|x_1 - x_2|^2 + |y_1 - y_2|^2)$$

and

$$\int_{|l|<c} |H(x_1, y_1, l) - H(x_2, y_2, l)|^\eta \nu(dl) \le K_2(|x_1 - x_2|^\eta + |y_1 - y_2|^\eta),$$

for $x_k, y_k \in R^d, k = 1, 2$.

The global Lipschitz condition (H4) implies the linear growth condition

$$|f(x, y)|^2 \vee |g(x, y)|^2 \le K_4(1 + |x|^2 + |y|^2)$$

and

$$\int_{|l|<c} |H(x, y, l)|^\eta \nu(dl) \le K_4(1 + |x|^\eta + |y|^\eta),$$

where $K_4 = 2^{\eta-1}(K_2 \vee |f(0, 0)|^2 \vee |g(0, 0)|^2 \vee \int_{|l|<c} |H(0, 0, l)|^\eta \nu(dl)), 2 \le \eta \le p$.

By Theorem 4.1, we get

Corollary 4.1 *Under hypothesis* (H1) *and the global Lipschitz condition* (H4)*, we have*

$$E\left[\sup_{0\le t\le T} |x(t) - y(t)|^2\right] \le C_6\Delta,$$

where C_6 is a positive constant independent of Δ.

Proof The proof is followed from the Corollary 5.1 in Mao [13] and is omitted here. □

5 Conclusion and outlook

In this paper, the numerical solutions of doubly perturbed stochastic delay differential equations driven by Lèvy process are considered. We define the numerical solutions and show that the numerical solutions converge to the true solutions under the local Lipschitz condition. Moreover, we give the order of convergence under the global Lipschitz condition. In our future works, we will investigate doubly perturbed stochastic delay differential equations with Markovian switching and doubly perturbed neutral SDDEs.

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